CONCLUSION

We have developed the basic properties of numerical ranges, have related the various numerical ranges and studied their relation to the spectrum.

We saw that the convex hull of the spectrum of an element of a unital Banach algebra could be approximated arbitrarily closely by the numerical range of that element defined with respect to some equivalent renorming of the algebra. From this we deduced a sharper version of a result by Williams [2] for operators over a Hilbert space. We also obtained a sufficient condition for an element to have as numerical range the convex hull of the spectrum and used this condition to prove that a normal type element has a numerical range equal to the convex hull of the spectrum.

We proved that an element of a unital Banach algebra is hermitian if and only if $[v(a + \lambda)]^2 \leq \rho[(a + \lambda)(a + \bar{\lambda})]$ for all complex numbers λ. Using this we characterized Banach*-algebras, in which every self-adjoint element is hermitian, as those algebras for which the inequality $v(a) \leq p(a) \equiv \sqrt{\rho}(aa^*)$ holds for every element a in the algebra.

From this we gave an elementary proof that such algebras are B*-algebras in the equivalent norm $p(\cdot)$. We then used a formula of L. Harris' to prove $p(\cdot) = \|\cdot\|$, thus giving a simple proof of

Palmer's characterization of B*-algebras among Banach algebras.

In Chapter 3 we investigated the closure properties of the spatial numerical range. We saw that a given normed linear space E could be embedded in a Banach space X with B(E) embedded in B(X) under the map $T \mapsto [T]$, such that $\overline{W(T)} \subseteq W([T]) \subsetneq V(T)$. We also saw that for a Hilbert space, or a class of spaces including the $\ell\rho$-spaces $(1 \leq \rho \leq \infty)$, the spatial numerical range of every compact operator was closed if and only if the space were finite dimensional.

We then proved that the spatial numerical range of a compact operator, on an $\ell\rho$-space $(1 < \rho < \infty)$ or a Hilbert space, contained all the non-zero extreme points of its closure. For a compact operator T on a Hilbert space, this showed that W(T) is closed if and only if $0 \in W(T)$. Several examples of compact operators T, on ℓ_2, with $0 \notin W(T)$ were then given to illustrate the exceptional behaviour of $\overline{W(T)} \setminus W(T)$ in this case.

We saw that for a class of reflexive Banach spaces, compact operators attained their numerical radius. We also saw that the hermitian operators which attain their numerical radius are dense among all the hermitian operators on a Hilbert space. From this we proved that for any operator T, on a Hilbert space, and $\varepsilon > 0$ there exists an operator T_1 such that $T - T_1$ is a rank one operator of norm less than ε and T_1 attains its norm. In this same chapter several areas for further investigation were indicated.